

SYMPLECTOMORPHISMS OF SURFACES PRESERVING A SMOOTH FUNCTION, I

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ABSTRACT. Let M be a compact orientable surface equipped with a volume form ω , P be either \mathbb{R} or S^1 , $f : M \rightarrow P$ be a C^∞ Morse map, and H be the Hamiltonian vector field of f with respect to ω . Let also $\mathcal{Z}_\omega(f) \subset C^\infty(M, \mathbb{R})$ be set of all functions taking constant values along orbits of H , and $\mathcal{S}_{\text{id}}(f, \omega)$ be the identity path component of the group of diffeomorphisms of M mutually preserving ω and f .

We construct a canonical map $\varphi : \mathcal{Z}_\omega(f) \rightarrow \mathcal{S}_{\text{id}}(f, \omega)$ being a homeomorphism whenever f has at least one saddle point, and an infinite cyclic covering otherwise. In particular, we obtain that $\mathcal{S}_{\text{id}}(f, \omega)$ is either contractible or homotopy equivalent to the circle.

Similar results hold in fact for a larger class of maps $M \rightarrow P$ whose singularities are equivalent to homogeneous polynomials without multiple factors.

1. INTRODUCTION

Let M be a closed oriented surface, $\text{Diff}(M)$ be the group of all C^∞ diffeomorphisms of M , and $\text{Diff}_0(M)$ be the identity path component of $\text{Diff}(M)$ consisting of all diffeomorphisms isotopic to the identity.

Let also $\text{Vol}(M, 1)$ be the space of all volume forms on M having volume 1 and $\omega \in \text{Vol}(M, 1)$. Since $\dim M = 2$, ω is a closed non-degenerate 2-form and so it defines a symplectic structure on M . Denote by $\text{Symp}(M, \omega)$ the group of all ω -preserving C^∞ diffeomorphisms, and let $\text{Symp}_0(M, \omega)$ be its identity path component.

Then Moser's stability theorem [20] implies that for any C^∞ family

$$\{\omega_t\}_{t \in D^n} \subset \text{Vol}(M, 1)$$

of volume forms parameterized by points of a closed n -dimensional disk D^n , there exists a C^∞ family of diffeomorphisms

$$\{h_t\}_{t \in D^n} \subset \text{Diff}_0(M)$$

such that $\omega_t = h_t^* \omega$ for all $t \in D^n$. In particular, this implies that the map

$$p : \text{Diff}_0(M) \rightarrow \text{Vol}(M, 1), \quad p(h) = h^* \omega$$

is a Serre fibration with fiber $\text{Symp}_0(M, \omega)$, see e.g. [19, §3.2], [2], or [21, §7.2].

Since $\text{Vol}(M, 1)$ is convex and therefore contractible, it follows from exact sequence of homotopy groups of the Serre fibration p that p yields isomorphisms of the corresponding homotopy groups $\pi_k \text{Symp}_0(M, \omega) \cong \pi_k \text{Diff}_0(M)$, $k \geq 0$. Hence the inclusion

$$(1.1) \quad \text{Symp}_0(M, \omega) \subset \text{Diff}_0(M)$$

turns out to be a weak homotopy equivalence. See also [18] for discussions of the inclusion (1.1) for non-compact manifolds.

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Moreover, let $\text{Diff}^+(M)$ be the group of orientation preserving diffeomorphisms. Then we have an inclusion $i : \text{Symp}(M, \omega) \subset \text{Diff}^+(M)$. Indeed, if h preserves ω , then it fixes the corresponding cohomology class $[\omega] \in H^2(M, \mathbb{R}) \cong \mathbb{R}$, and so yields the identity on $H^2(M, \mathbb{R})$. In particular, h preserves orientation of M . Hence (1.1) also implies that i yields a monomorphism $i_0 : \pi_0 \text{Symp}(M, \omega) \rightarrow \pi_0 \text{Diff}^+(M)$ on the set of isotopy classes.

It is well known that $\pi_0 \text{Diff}^+(M)$ is generated by isotopy classes of Dehn twists, [4], [9], and one easily shows that each Dehn twist can be realized by ω -preserving diffeomorphism. This implies that i_0 is also surjective, and so i is a weak homotopy equivalence as well.

On the other hand, let $f : M \rightarrow \mathbb{R}$ be a Morse function,

$$\text{Stab}(f) = \{h \in \text{Diff}(M) \mid f \circ h = f\}$$

be the group of f -preserving diffeomorphisms, i.e. the *stabilizer* of f with respect to the right action of $\text{Diff}(M)$ on $C^\infty(M, \mathbb{R})$, and $\text{Stab}_0(f)$ be its identity path component. Let also

$$\mathcal{O}(f) = \{f \circ h \mid h \in \text{Diff}(M)\}$$

be the corresponding *orbit* of f ,

$$\text{Stab}(f, \omega) = \text{Stab}(f) \cap \text{Symp}(M, \omega)$$

be the group of diffeomorphisms mutually preserving f and ω , and $\text{Stab}_0(f, \omega)$ be its identity path component.

In a series of papers the author proved that $\text{Stab}_0(f)$ is either *contractible* or *homotopy equivalent to the circle* and computed the higher homotopy groups of $\mathcal{O}(f)$, [11], [15]; showed that $\mathcal{O}(f)$ is homotopy equivalent to a finite-dimensional CW-complex, [12]; and recently described precise algebraic structure of the fundamental group $\pi_1 \mathcal{O}(f)$, [17]. E. Kudryavtseva, [7], [8], studied the homotopy type of the space of Morse maps on compact surfaces and using similar ideas as in [11], [15] proved that $\mathcal{O}(f)$ has the homotopy type of a quotient of a torus by a free action of a certain finite group.

The present paper is former in a series subsequent ones devoted to extension of the above results to the case of ω -preserving diffeomorphisms. We will describe here the homotopy type of $\text{Stab}_0(f, \omega)$. In next papers will study the homotopy type of the subgroup of $\text{Stab}(f, \omega)$ trivially acting on the Kronrod-Reeb graph of f , see §3.2, and describe the precise algebraic structure of $\pi_0 \text{Stab}(f, \omega)$.

Notice that if H is the Hamiltonian vector field of f and $\mathbf{H} : M \times \mathbb{R} \rightarrow M$ is the corresponding Hamiltonian flow, then $\mathbf{H}_t \in \text{Stab}(f, \omega)$ for all $t \in \mathbb{R}$.

More generally, given a C^∞ function $\alpha : M \rightarrow \mathbb{R}$, one can define the map

$$\mathbf{H}_\alpha : M \rightarrow M, \quad \mathbf{H}_\alpha(x) = \mathbf{H}(x, \alpha(x)),$$

being in general just a C^∞ map leaving invariant each orbit of H , and so preserving f . However, \mathbf{H}_α is not necessarily a diffeomorphism.

Let $\mathcal{Z}(f) = \{\alpha \in C^\infty(M, \mathbb{R}) \mid H(\alpha) = 0\}$ be the algebra of all smooth functions taking constant values along orbits of \mathbf{H} . Equivalently, $\mathcal{Z}(f)$ is the *centralizer* of f with respect to the Poisson bracket induced by ω , see §2.3. In Lemma 3.2.1 we also identify $\mathcal{Z}(f)$ with a certain subset of continuous functions on the Kronrod-Reeb graph of f . In particular, $\mathcal{Z}(f)$ contains all constant functions.

We will prove in Theorem 3.0.3 that $\mathbf{H}_\alpha \in \text{Stab}_0(f, \omega)$ if and only if $\alpha \in \mathcal{Z}(f)$. Moreover if f has at least one saddle critical point, then the correspondence $\alpha \mapsto \mathbf{H}_\alpha$ is a homeomorphism

$\mathcal{Z}(f) \cong \text{Stab}_0(f, \omega)$ with respect to C^∞ topologies, and so $\text{Stab}_0(f, \omega)$ is contractible. Otherwise, that correspondence is an infinite cyclic covering map and $\text{Stab}_0(f, \omega)$ is homotopy equivalent to the circle. It will also follow that the inclusion

$$\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$$

is a homotopy equivalence. This statement can be regarded as an analogue of (1.1) for f -preserving diffeomorphisms.

Again it implies that the inclusion

$$j : \text{Stab}(f, \omega) \subset \text{Stab}^+(f) \equiv \text{Stab}(f) \cap \text{Diff}^+(M)$$

yields an injection $j_0 : \pi_0 \text{Stab}(f, \omega) \rightarrow \pi_0 \text{Stab}^+(f)$ on the sets of isotopy classes. However, now j_0 is not necessarily surjective, see §3.3. The reason is that $\text{Stab}^+(f)$ has many invariant subsets, e.g. the sets of the form $M_a = f^{-1}(-\infty, a]$, $a \in \mathbb{R}$, and so if $h \in \text{Stab}(f, \omega)$ interchanges connected components of M_a , then they must have the same ω -volume.

In fact, our results hold for a larger class of smooth maps f from M into \mathbb{R} and S^1 , see §2.4. On the other hand, we also provide in §3.1 an example of a function with isolated critical points for which the above correspondence $\alpha \mapsto \mathbf{H}_\alpha$ is not surjective.

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2. PRELIMINARIES

2.1. Shift map. Let M be a connected n -dimensional C^∞ manifold, H be a C^∞ vector field tangent to ∂M and generating a flow $\mathbf{H} : M \times \mathbb{R} \rightarrow M$. For each $\alpha \in C^\infty(M, \mathbb{R})$ define the following C^∞ map $\mathbf{H}_\alpha : M \rightarrow M$ by

$$\mathbf{H}_\alpha(x) = \mathbf{H}(x, \alpha(x)),$$

for $x \in M$. Evidently, \mathbf{H}_α leaves invariant each orbit of \mathbf{H} and is homotopic to id_M by the homotopy $\{\mathbf{H}_{t\alpha}\}_{t \in [0,1]}$. Also notice that if $\alpha \equiv t$ is a constant function, then $\mathbf{H}_\alpha = \mathbf{H}_t$ is a diffeomorphism belonging to the flow \mathbf{H} .

For $\alpha \in C^\infty(M, \mathbb{R})$ we will denote by $H(\alpha)$ the Lie derivative of α along H .

2.1.1. Lemma. [10, Theorem 19] *Let $\alpha \in C^\infty(M, \mathbb{R})$, $y \in M$, and $z = \mathbf{H}_\alpha(y)$. Then the tangent map $T_y \mathbf{H}_\alpha : T_y M \rightarrow T_z M$ is an isomorphism if and only if $1 + H(\alpha)(y) \neq 0$.*

2.1.2. Remark. In fact, [10, Lemma 20], if $\alpha(y) = 0$, so $z = \mathbf{H}_\alpha(y) = \mathbf{H}(y, 0) = y$ is a fixed point of \mathbf{H}_α , then the determinant of $T_y \mathbf{H}_\alpha : T_y M \rightarrow T_y M$ does not depend on a particular choice of local coordinates at z and equals $1 + H(\alpha)(y)$. The general case $\alpha(y) = a \neq 0$ reduces to $a = 0$ by observation that $\mathbf{H}_\alpha = \mathbf{H}_{\alpha-a} \circ \mathbf{H}_a$.

To get a global variant of Lemma 2.1.1 notice that the correspondence $\alpha \mapsto \mathbf{H}_\alpha$ can also be regarded as the following mapping

$$\varphi_H : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, M), \quad \varphi_H(\alpha) = \mathbf{H}_\alpha.$$

It will be called the *shift map* along orbits of \mathbf{H} , [10], [16]. Consider the following subset of $C^\infty(M, \mathbb{R})$:

$$(2.1) \quad \Gamma_H = \{\alpha \in C^\infty(M, \mathbb{R}) \mid 1 + H(\alpha) > 0\},$$

and let $\text{Diff}_0(H)$ be the group of all diffeomorphisms of M which leave invariant each orbit of H and isotopic to the identity via an orbit preserving isotopy.

2.1.3. **Lemma.** [10, Theorem 19] *If M is compact, then*

$$(2.2) \quad \varphi(\Gamma_H) \subset \text{Diff}_0(H), \quad \Gamma_H = \varphi^{-1}(\text{Diff}_0(H)).$$

In other words, suppose $\alpha \in C^\infty(M, \mathbb{R})$. Then $\alpha \in \Gamma$ if and only if $\mathbf{H}_\alpha \in \text{Diff}_0(H)$.

2.2. **Hamiltonian vector field.** Let M be a compact orientable surface equipped with a volume form ω and P be either \mathbb{R} or S^1 . Since $\dim M = 2$, ω is a closed 2-form, and therefore it defines a symplectic structure on M . Then for each C^1 map $f : M \rightarrow P$ there exists a unique vector field H on M satisfying

$$(2.3) \quad df(z)(u) = \omega(u, H(z)),$$

for each point $z \in M$ and a tangent vector $u \in T_z M$. This vector field is called the *Hamiltonian* vector field of f with respect to ω . For the convenience of the reader we recall its construction as it is usually defined for functions $f : M \rightarrow \mathbb{R}$ only.

Let $z \in M$. Fix local charts $h : U \rightarrow M$ and $q : J \rightarrow P$ at z and $f(z)$ respectively, where U is an open subset of the upper half-plane $\mathbb{R}_+^2 = \{(x, y) \mid y \geq 0\}$ and J is an open interval in \mathbb{R} . Decreasing U one can assume that $f(h(U)) \subset q(J)$. Then the map $\hat{f} = q^{-1} \circ f \circ h : U \rightarrow J$ is called a *local representation* of f at z .

Now if in coordinates (x, y) on U we have that $\omega(x, y) = \gamma(x, y)dx \wedge dy$ for some non-zero C^∞ function $\gamma : U \rightarrow \mathbb{R} \setminus \{0\}$, then

$$(2.4) \quad H(x, y) = \frac{1}{\gamma(x, y)} \left(-\hat{f}'_y \frac{\partial}{\partial x} + \hat{f}'_x \frac{\partial}{\partial y} \right).$$

A definition of H that does not use local coordinates can be given as follows. Since the restriction of ω to each tangent space $T_x M$ is a non-degenerate skew-symmetric form, it follows that ω yields a bundle isomorphism

$$\begin{array}{ccc} TM & \xrightarrow{\psi} & T^*M \\ & \searrow & \swarrow \\ & M & \end{array}$$

defined by the formula $\psi(u)(v) = \omega(u, v)$ for all $u, v \in T_x M$ and $x \in M$.

Further notice, that the tangent bundle of P is trivial, so we have the *unit* section

$$s : P \rightarrow TP \equiv P \times \mathbb{R}, \quad s(q) = (q, 1).$$

Now for a C^1 map $f : M \rightarrow P$ its *differential* $df : M \rightarrow T^*M$ and the *Hamiltonian* vector field $H : M \rightarrow TM$ are unique maps for which the following diagram is commutative:

$$\begin{array}{ccccc} TM & \xrightarrow{\psi} & T^*M & \xleftarrow{T^*f} & T^*P \equiv P \times \mathbb{R} \\ & \searrow H & \uparrow df & & \uparrow s \\ & & M & \xrightarrow{f} & P \end{array}$$

Thus $df = T^*f \circ s \circ f$, and $H = \psi^{-1} \circ df$. It follows that

$$(2.5) \quad H(z)(f) = \omega(H(z), H(z)) = 0,$$

as ω is skew-symmetric, and so H is *tangent to level curves of f* .

Suppose, in addition, that f takes constant values at boundary components of M . Then, due to (2.5), H is tangent to ∂M , and therefore it yields a flow $\mathbf{H} : M \times \mathbb{R} \rightarrow M$. It also

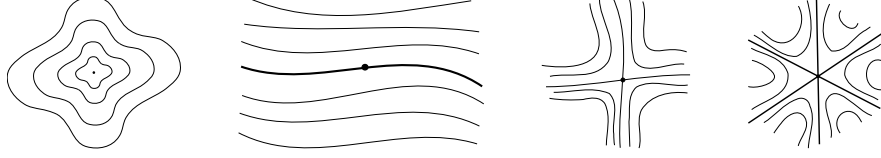


FIGURE 2.1. Level sets foliation for singularities of Axiom (L)

follows from (2.5) that each diffeomorphism $\mathbf{H}_t : M \rightarrow M$ preserves f , in the sense that $f \circ \mathbf{H}_t = f$. Moreover, the well known Liouville's theorem claims that each diffeomorphism \mathbf{H}_t also preserves ω . In fact, that theorem is a simple consequence of Cartan's identity:

$$(2.6) \quad \mathcal{L}_H \omega = d(\iota_H \omega) + \iota_H d\omega = d(df) + \iota_H 0 = 0,$$

since $\iota_H \omega = \omega(H, \cdot) = df$ by (2.3), and $d\omega = 0$ as $\dim \omega = \dim M$.

2.3. Poisson multiplication. Let Q be another one-dimensional manifold without boundary, so Q is either \mathbb{R} or S^1 as well as P . Then ω yields a *Poisson* multiplication

$$(2.7) \quad \{\cdot, \cdot\} : C^\infty(M, P) \times C^\infty(M, Q) \longrightarrow C^\infty(M, \mathbb{R})$$

defined by one of the following equivalent formulas:

$$(2.8) \quad \{f, g\} := \omega(H_f, H_g) = \psi(H_f)(H_g) = H_f(g) = -H_g(f),$$

where H_f and H_g are Hamiltonian vector fields of $f \in C^\infty(M, P)$ and $g \in C^\infty(M, Q)$ respectively.

In particular, for each $f \in C^\infty(M, P)$ one can define its *annulator* with respect to (2.8) by

$$(2.9) \quad \mathcal{Z}_\omega^Q(f) = \{g \in C^\infty(M, Q) \mid H_f(g) = \{f, g\} = 0\}.$$

Thus $\mathcal{Z}_\omega^Q(f)$ consists of all maps $g \in C^\infty(M, Q)$ taking constant values along orbits of the Hamiltonian vector field H_f . It follows from (2.8) that $g \in \mathcal{Z}_\omega^Q(f)$ iff $f \in \mathcal{Z}_\omega^P(g)$.

When $P = Q = \mathbb{R}$, this multiplication is the usual *Poisson bracket*, and $\mathcal{Z}_\omega^\mathbb{R}(f)$ is the *centralizer* of f , see [19, §3].

2.4. Class $\mathcal{F}(M, P)$. Let $\mathcal{F}(M, P)$ be the subspace of $C^\infty(M, P)$ consisting of maps f satisfying the following two axioms:

Axiom (B) *The map f takes a constant value at each connected component of ∂M and has no critical points on ∂M .*

Axiom (L) *For every critical point z of f there is a local presentation $\hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}$ of f near z in which \hat{f} is a homogeneous polynomial $\mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple factors.*

In particular, since the polynomial $\pm x^2 \pm y^2$ (a non-degenerate singularity) is homogeneous and has no multiple factors, we see that $\mathcal{F}(M, P)$ contains an open and everywhere dense subset $\text{Morse}(M, P)$ consisting of maps satisfying Axiom (B) and having non-degenerate critical points only.

Figure 2.1 describes possible singularities satisfying Axiom (L).

2.4.1. Definition. *We will say that a vector field F on M is **Hamiltonian like** for $f \in \mathcal{F}(M, P)$ if*

- (a) $F(f) = 0$, and, in particular, F is tangent to ∂M and generates a flow on M ;
- (b) $F(z) = 0$ if and only if z is a critical point of f ;

- (c) for each z critical point of f there exists a local representation $\hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}$ of f as a homogeneous polynomial without multiple factors such that in these coordinates $F(x, y) = -\hat{f}'_y \frac{\partial}{\partial x} + \hat{f}'_x \frac{\partial}{\partial y}$.

One can easily prove that for each $f \in \mathcal{F}(M, P)$ there exists a Hamiltonian like vector field, [11, Lemma 5.1].

Notice also that every Hamiltonian vector field H of f has properties (a) and (b) of Definition 2.4.1. Moreover, if H is also a Hamiltonian like, then due to (2.4) in the corresponding coordinates satisfying property (c) of Definition 2.4.1 we have that $\omega = dx \wedge dy$.

2.4.2. Lemma. *Let F be any Hamiltonian like vector field for $f \in \mathcal{F}(M, P)$, and H be the Hamiltonian vector field for f with respect to ω . Then there exists an everywhere non-zero C^∞ function $\lambda : M \rightarrow \mathbb{R} \setminus \{0\}$ such that $H = \lambda F$.*

Proof. Denote by Σ_f the set of critical point of f , being also the set of zeros of H as well as of F . Since F and H are parallel and non-zero on $M \setminus \Sigma_f$, it follows that there exists a C^∞ non-zero function $\lambda : M \setminus \Sigma_f \rightarrow \mathbb{R}$ such that $H = \lambda F$. It remains to show that λ can be defined by non-zero values on Σ_f to give a C^∞ function on all of M .

Let z be a critical point of f . Then by definition of Hamiltonian like vector field there exists a local representation $\hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}$ of f such that $z = (0, 0) \in \mathbb{R}^2$, \hat{f} is a homogeneous polynomial without multiple factors, and $F = -\hat{f}'_y \frac{\partial}{\partial x} + \hat{f}'_x \frac{\partial}{\partial y}$.

Then $\omega(x, y) = \gamma(x, y)dx \wedge dy$ for some non-zero C^∞ function γ , and by formula (2.4), we have $H = \frac{1}{\gamma}F$ on $U \setminus z$. Hence $\lambda = 1/\gamma$, and so λ smoothly extends to all of U by $\lambda(z) = 1/\gamma(z)$. \square

The following statement is a particular case of results of [13] on parameter rigidity.

2.4.3. Corollary. *c.f. [13, §4 & Theorem 11.1] For any two Hamiltonian like vector fields F_1 and F_2 there exists an everywhere non-zero C^∞ function $\mu : M \rightarrow \mathbb{R} \setminus \{0\}$ such that $F_1 = \mu F_2$.*

Proof. It follows from Lemma 2.4.2 that $H = \lambda_1 F_1 = \lambda_2 F_2$ for some everywhere non-zero C^∞ functions $\lambda_1, \lambda_2 : M \rightarrow \mathbb{R} \setminus \{0\}$. Hence $\mu = \lambda_2/\lambda_1$. \square

2.5. Topological type of $\text{Stab}_0(f)$. Let $f \in \mathcal{F}(M, P)$, H be a Hamiltonian like vector field for f , and $\mathbf{H} : M \times \mathbb{R} \rightarrow M$ be the corresponding Hamiltonian flow.

2.5.1. Theorem. [14], [15], [16]. *Let $\varphi : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, M)$ be the shift map along orbits of H and*

$$\Gamma = \{\alpha \in C^\infty(M, \mathbb{R}) \mid 1 + H(\alpha) > 0\},$$

see (2.1).

- (1) $\varphi(\Gamma) = \text{Stab}_0(f)$ and $\Gamma = \varphi^{-1}(\text{Stab}_0(f))$.
- (2) Suppose all critical points of f are **non-degenerate local extremes**, so, in particular, $f \in \text{Morse}(M, P)$. Then the restriction map $\varphi|_\Gamma : \Gamma \rightarrow \text{Stab}_0(f)$ is an infinite cyclic covering, and so $\text{Stab}_0(f)$ is homotopy equivalent to the circle. More precisely, in this case there exists $\theta \in \Gamma$ such that
 - (i) $\theta > 0$ on all of M ;
 - (ii) each non-constant orbit γ of F is periodic, and θ takes a constant value on γ being an positive integral multiple of the period $\text{Per}(\gamma)$ of γ ;

- (iii) *there exists a free action of \mathbb{Z} on Γ defined by $n * \alpha = \alpha + n\theta$, for $n \in \mathbb{Z}$ and $\alpha \in \Gamma$, such that the map φ is a composite*

$$(2.10) \quad \varphi : \Gamma \xrightarrow{p} \Gamma/\mathbb{Z} \xrightarrow[\cong]{r} \text{Stab}_0(f),$$

where p is a projection onto the factor space Γ/\mathbb{Z} endowed with the corresponding final topology, and r is a homeomorphism.

- (3) *Suppose f has a critical point being **not a non-degenerate local extreme**. Then $\varphi|_\Gamma : \Gamma \rightarrow \text{Stab}_0(f)$ is a homeomorphism, and so $\text{Stab}_0(f)$ is contractible.*

Proof. In fact, Theorem 2.5.1 is stated and proved in [15] for any *Hamiltonian like* vector field F of f . The advantage of using Hamiltonian like vector fields is that we have *precise* formulas for F near critical points of f .

Let $\lambda : M \rightarrow \mathbb{R}$ be every where non-zero C^∞ function and $H = \lambda F$. We will deduce from results of [14] that Theorem 2.5.1 also holds for H . Due to Lemma 2.4.2 this includes the case when H is Hamiltonian.

Let $\mathbf{F}, \mathbf{H} : M \times \mathbb{R} \rightarrow M$ be the flows of F and $H = \lambda F$ respectively,

$$\varphi_F, \varphi_H : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, M)$$

be the corresponding shift maps, $\text{image}(\varphi_H)$, $\text{image}(\varphi_F)$ be their images in $C^\infty(M, M)$, and Γ_F, Γ_H be corresponding the subsets of $C^\infty(M, \mathbb{R})$ defined by (2.1). Define the following C^∞ function

$$\sigma : M \times \mathbb{R} \rightarrow \mathbb{R}, \quad \sigma(x, s) = \int_0^s \lambda(H(x, t)) dt.$$

Then it is well known and easy to see, e.g. [14], that for each $\alpha \in C^\infty(M, \mathbb{R})$ we have that

$$(2.11) \quad \mathbf{H}(x, \alpha(x)) = \mathbf{F}(x, \sigma(x, \alpha(x))).$$

Consider the map

$$\gamma : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R}), \quad \gamma(\alpha)(x) = \sigma(x, \alpha(x)).$$

Evidently, γ is continuous with respect to C^∞ topologies. Moreover, (2.11) means that $\mathbf{H}_\alpha = \mathbf{F}_{\gamma(\alpha)}$ for all $\alpha \in C^\infty(M, \mathbb{R})$. Hence $\varphi_H = \varphi_F \circ \gamma$, $\text{image}(\varphi_H) \subset \text{image}(\varphi_F)$, and we get the following commutative diagram:

$$\begin{array}{ccccccc} \Gamma_H & \hookrightarrow & C^\infty(M, \mathbb{R}) & \xrightarrow{\varphi_H} & \text{image}(\varphi_H) & \hookrightarrow & C^\infty(M, M) \\ & & \downarrow \gamma & & \downarrow & & \\ \Gamma_F & \hookrightarrow & C^\infty(M, \mathbb{R}) & \xrightarrow{\varphi_F} & \text{image}(\varphi_F) & \hookrightarrow & C^\infty(M, M) \end{array}$$

Since $\lambda \neq 0$ everywhere, one can interchange $F = \frac{1}{\lambda}H$ and H . Hence by the same arguments as above we get that $\text{image}(\varphi_H) = \text{image}(\varphi_F)$ and γ is a homeomorphism. Also notice that the orbit structures of F and H coincide. Hence $\text{Diff}_0(F) = \text{Diff}_0(H)$, and so

$$\begin{aligned} \Gamma_H &\stackrel{(2.2)}{=} \varphi_H^{-1}(\text{image}(\varphi_H) \cap \text{Diff}_0(H)) = \varphi_H^{-1}(\text{image}(\varphi_F) \cap \text{Diff}_0(F)) \\ &= \gamma^{-1} \circ \varphi_F^{-1}(\text{image}(\varphi_F) \cap \text{Diff}_0(F)) \stackrel{(2.2)}{=} \gamma^{-1}(\Gamma_F). \end{aligned}$$

Thus γ yields a homeomorphism of Γ_H onto Γ_F . Since Theorem 2.5.1 holds for F , we get the following commutative diagram

$$\begin{array}{ccc} \Gamma_H & \xrightarrow[\cong]{\gamma} & \Gamma_F \\ & \searrow \varphi_H \quad \swarrow \varphi_F & \\ & \text{Stab}_0(f) & \end{array}$$

implying that $\varphi_H|_{\Gamma_H}$ has the same topological properties (1)-(3) as $\varphi_F|_{\Gamma_F}$, and so Theorem 2.5.1 holds for H as well. \square

2.5.2. Remark. Let us discuss the case (2) of Theorem 2.5.1 which is realized precisely for the following four types of Morse maps, see [11, Theorem 1.9]:

- (A) $M = S^2$ is a 2-sphere and $f : S^2 \rightarrow P$ has exactly two critical points: non-degenerate local minimum and maximum;
- (B) $M = D^2$ is a 2-disk and $f : D^2 \rightarrow P$ has exactly one critical point being a non-degenerate local extreme;
- (C) $M = S^1 \times [0, 1]$ is a cylinder and $f : S^1 \times [0, 1] \rightarrow P$ has no critical points;
- (D) $M = T^2$ is a 2-torus, $P = S^1$ is a circle, and $f : T^2 \rightarrow P$ has no critical points.

Due to (i) and (ii) each regular point $x \in M$ of f is periodic of some period $\text{Per}(x)$, and there exists $k_x \in \mathbb{N}$ depending on x such that $\theta(x) = k_x \text{Per}(x)$. Hence

$$\mathbf{H}_\theta(x) = \mathbf{H}(x, \theta(x)) = \mathbf{H}(x, k_x \text{Per}(\gamma)) = x,$$

and so $\mathbf{H}_\theta = \text{id}_M$. Moreover, if $\alpha \in \Gamma$, then

$$\begin{aligned} \mathbf{H}_{\alpha+n\theta}(x) &= \mathbf{H}(x, \alpha(x) + n\theta(x)) = \mathbf{H}(\mathbf{H}(x, n\theta(x)), \alpha(x)) = \\ &= \mathbf{H}(x, \alpha(x)) = \mathbf{H}_\alpha(x). \end{aligned}$$

This implies correctness of the \mathbb{Z} -action from (iii) of Theorem 2.5.1 and existence of decomposition (2.10) with continuous p and r . The principal difficulty was to prove that r is a homeomorphism.

The aim of the present paper is to deduce from Theorem 2.5.1 a description of the homotopy type of $\text{Stab}_0(f, \omega)$, see Theorem 3.0.3 below.

3. MAIN RESULT

Let M be a compact orientable surface equipped with a volume form ω , $f \in \mathcal{F}(M, P)$, H be the Hamiltonian vector field of f with respect to ω , $\mathbf{H} : M \times \mathbb{R} \rightarrow M$ be the corresponding Hamiltonian flow, and

$$\varphi : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, M), \quad \varphi(\alpha)(x) = \mathbf{H}(x, \alpha(x))$$

be the shift map along orbits of \mathbf{H} . Let also

$$\mathcal{Z} = \mathcal{Z}_\omega^\mathbb{R}(f) = \{\alpha \in C^\infty(M, \mathbb{R}) \mid H(\alpha) = 0\}$$

be the space of functions taking constant values along orbits of H , see (2.9). Then \mathcal{Z} is a linear subspace of $C^\infty(M, \mathbb{R})$ and is contained in Γ , see (2.1). In particular, \mathcal{Z} is contractible as well as Γ .

3.0.3. Theorem. *The following statements hold true.*

- (1) $\varphi(\mathcal{Z}) = \text{Stab}_0(f, \omega) = \text{Stab}_0(f) \cap \text{Stab}(f, \omega)$ and $\mathcal{Z} = \varphi^{-1}(\text{Stab}_0(f, \omega))$.

- (2) If all critical points of f are non-degenerate local extremes, then the restriction $\varphi|_{\mathcal{Z}} : \mathcal{Z} \rightarrow \text{Stab}_0(f, \omega)$ is an infinite cyclic covering, and $\text{Stab}_0(f, \omega)$ is homotopy equivalent to the circle.
- (3) Otherwise, $\varphi|_{\mathcal{Z}} : \mathcal{Z} \rightarrow \text{Stab}_0(f, \omega)$ is a homeomorphism, and so $\text{Stab}_0(f, \omega)$ is contractible.
- (4) The inclusion $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ is a homotopy equivalence.
- (5) The inclusion map $j : \text{Stab}(f, \omega) \subset \text{Stab}(f)$ induces an injection $j_0 : \pi_0 \text{Stab}(f, \omega) \rightarrow \pi_0 \text{Stab}(f)$.

Proof. First we need the following lemma.

3.0.4. Lemma. *Let $\alpha \in \mathcal{C}^\infty(M, \mathbb{R})$. Then the action of \mathbf{H}_α on ω is given by*

$$(3.1) \quad \mathbf{H}_\alpha^* \omega = (1 + H(\alpha)) \cdot \omega.$$

Proof. Since the set of critical points is finite and so nowhere dense, it suffices to check this relation at regular points of f only.

So let p be a regular point of f . Then $H(p) \neq 0$, whence there are local coordinates (x, y) at p in which $p = (0, 0)$, $H(x, y) = \frac{\partial}{\partial x}$, and $\mathbf{H}(x, y, t) = (x + t, y)$ for sufficiently small x, y, t . In particular, $H(\alpha) = \frac{\partial \alpha}{\partial x}$. We also have that $\omega(x, y) = \gamma(x, y) dx \wedge dy$ for some C^∞ function γ .

Notice that one may also assume that $\alpha(p) = 0$. Indeed, let $b = \alpha(p)$. Then $\mathbf{H}_\alpha = \mathbf{H}_{\alpha-b} \circ \mathbf{H}_b$. Since \mathbf{H}_b preserves ω , see (2.6), it follows that

$$\mathbf{H}_\alpha^* \omega = \mathbf{H}_{\alpha-b}^* \circ \mathbf{H}_b^* \omega = \mathbf{H}_{\alpha-b}^* \omega.$$

Thus suppose $\alpha(p) = 0$, whence $\mathbf{H}_\alpha(p) = p$. Then

$$\begin{aligned} \mathbf{H}_\alpha^* \omega(x, y) &= \gamma \circ \mathbf{H}_\alpha(x, y) d(x + \alpha) \wedge dy \\ &= \gamma(x + \alpha, y) (1 + \alpha'_x) dx \wedge dy \\ &= \gamma(x + \alpha, y) (1 + H(\alpha)) dx \wedge dy. \end{aligned}$$

In particular, at p we have that

$$\mathbf{H}_\alpha^* \omega(p) = (1 + H(\alpha)(p)) \cdot \omega(p),$$

which proves (3.1). □

Now we can complete Theorem 3.0.3.

(1) Let us check that

$$(3.2) \quad \varphi(\mathcal{Z}) \subset \text{Stab}_0(f, \omega).$$

Let $\alpha \in \mathcal{Z}$. As \mathbf{H}_α leaves invariant each orbit of H , and therefore it preserves f , we have that $\mathbf{H}_\alpha \in \text{Stab}(f)$.

Moreover, by formula (3.1), $\mathbf{H}_\alpha^* \omega = \omega$, so $\mathbf{H}_\alpha \in \text{Stab}(f, \omega)$.

Now notice that $t\alpha \in \mathcal{Z}$ for all $t \in \mathbb{R}$, and so $\mathbf{H}_{t\alpha} \in \text{Stab}(f, \omega)$ as well. Thus the homotopy $\mathbf{H}_{t\alpha} : M \rightarrow M$, $t \in [0, 1]$, is in fact an isotopy in $\text{Stab}(f, \omega)$ between $\text{id}_M = \mathbf{H}_0$ and \mathbf{H}_α . Hence $\mathbf{H}_\alpha \in \text{Stab}_0(f, \omega)$.

Further we claim that

$$(3.3) \quad \mathcal{Z} \supset \varphi^{-1}(\text{Stab}_0(f) \cap \text{Stab}(f, \omega)).$$

Indeed, let $h \in \text{Stab}_0(f) \cap \text{Stab}(f, \omega)$. Then by (1) of Theorem 2.5.1 $h = \mathbf{H}_\alpha$ for some $\alpha \in \Gamma$. As ω is everywhere non-zero on M , it follows from formula (3.1) that $H(\alpha) = 0$ on all of M , that is $\alpha \in \mathcal{Z}$.

Hence

$$\begin{aligned} \varphi(\mathcal{Z}) &\stackrel{(3.2)}{\subset} \text{Stab}_0(f, \omega) \subset \text{Stab}_0(f) \cap \text{Stab}(f, \omega) \stackrel{(3.3)}{\subset} \varphi(\mathcal{Z}), \\ \mathcal{Z} &\stackrel{(3.2)}{\subset} \varphi^{-1}(\text{Stab}_0(f, \omega)) \subset \varphi^{-1}(\text{Stab}_0(f) \cap \text{Stab}(f, \omega)) \stackrel{(3.3)}{\subset} \mathcal{Z}. \end{aligned}$$

This proves (1).

(2), (4) Suppose $\varphi : \Gamma \rightarrow \text{Stab}_0(f)$ is an infinite cyclic covering map, and let $\theta \in \Gamma$ be the function from (2) of Theorem 2.5.1.

Then due to property (ii) in Theorem 2.5.1 θ takes constant values along orbits of H , and therefore $\theta \in \mathcal{Z}$. Since, in addition, \mathcal{Z} is a group, it follows that \mathcal{Z} is invariant with respect to the \mathbb{Z} -action on Γ , i.e. $\alpha + n\theta \in \mathcal{Z}$ for all $\alpha \in \mathcal{Z}$. Therefore $\mathcal{Z} = \varphi^{-1}(\text{Stab}_0(f, \omega))$. Hence $\varphi|_{\mathcal{Z}} : \mathcal{Z} \rightarrow \text{Stab}_0(f, \omega)$ is a \mathbb{Z} -covering as well as $\varphi|_{\Gamma}$. As \mathcal{Z} is contractible, we obtain that the quotient $\text{Stab}_0(f, \omega)$ is homotopy equivalent to the circle.

Consider the following path $\tau : [0, 1] \rightarrow \mathcal{Z} \subset \Gamma$, $\tau(t) = t\theta$. Then $\varphi \circ \tau$ is a loop in $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$, since

$$\varphi \circ \tau(1)(x) = \mathbf{F}(x, \theta(x)) = x = \mathbf{F}(x, 0) = \varphi \circ \tau(0)(x).$$

This loop is a generator of $\pi_1 \text{Stab}_0(f, \omega) \cong \mathbb{Z}$ as well as a generator of $\pi_1 \text{Stab}_0(f) \cong \mathbb{Z}$. Hence the inclusion $j : \text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ yields an isomorphism of fundamental groups. Since these spaces homotopy equivalent to the circle, we obtain that j is a homotopy equivalence.

(3), (4) If $\varphi : \Gamma \rightarrow \text{Stab}_0(f)$ is a homeomorphism, then due to (1) it yields a homeomorphism of \mathcal{Z} onto $\text{Stab}_0(f, \omega)$. In particular, both $\text{Stab}_0(f, \omega)$ and $\text{Stab}_0(f)$ are contractible, and so the inclusion $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ is a homotopy equivalence.

(5) Injectivity of j_0 follows from the relation $\text{Stab}_0(f, \omega) = \text{Stab}_0(f) \cap \text{Stab}(f, \omega)$. Theorem 3.0.3 is completed. \square

3.0.5. Remark. Though the inclusion $\text{Stab}_0(f, \omega) \subset \text{Stab}_0(f)$ is a homotopy equivalence, it seems to be a highly non-trivial task to find precise formulas for a *strong* deformation retraction of $\text{Stab}_0(f)$ onto $\text{Stab}_0(f, \omega)$. For the case (3) of Theorem 3.0.3 this is equivalent to a construction of a strong deformation retraction of Γ onto \mathcal{Z} . In fact, it suffices to find a retraction $r : \Gamma \rightarrow \mathcal{Z}$, so to associate to each $\alpha \in \Gamma$ a function $r(\alpha)$ taking constant values along orbits of H so that each $\beta \in \mathcal{Z}$ remains unchanged. Then a strong deformation $r_t : \Gamma \rightarrow \mathcal{Z}$, $t \in [0, 1]$, of Γ onto \mathcal{Z} can be given by $r_t(\alpha) = (1 - t)\alpha + tr(\alpha)$.

3.1. Counterexample for maps $g \notin \mathcal{F}(M, P)$. Let $D^2 = \{|z| \leq 1\}$ be the unit disk in the complex plane \mathbb{C} and $\omega = dx \wedge dy$ be the standard symplectic form. Consider the following two functions $f, g : D^2 \rightarrow [0, 1]$ defined by

$$f(x, y) = x^2 + y^2 = |z|^2, \quad g(x, y) = (x^2 + y^2)^2 = |z|^4.$$

Then the foliations by level sets of f and g coincide, whence

$$\mathcal{Z}_\omega^\mathbb{R}(f) = \mathcal{Z}_\omega^\mathbb{R}(g), \quad \text{Stab}_0(f, \omega) = \text{Stab}_0(g, \omega), \quad \text{Stab}_0(f) = \text{Stab}_0(g).$$

However, $f \in \mathcal{F}(D^2, \mathbb{R})$, while g does not belong to $\mathcal{F}(D^2, \mathbb{R})$ since it is a polynomial with multiple factors.

Notice that the Hamiltonian vector fields F and G of f and g are given by

$$F(x, y) = -2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}, \quad G(x, y) = 2(x^2 + y^2)F(x, y).$$

In particular, the Hamiltonian flow $\mathbf{F} : D^2 \times \mathbb{R} \rightarrow D^2$ of f is given by $\mathbf{F}(z, t) = e^{2it}z$, and so the tangent map $T_0\mathbf{F}_t : T_0D^2 \rightarrow T_0D^2$ is not the identity for $t \neq \pi n$, $n \in \mathbb{Z}$.

On the other hand, the linear part of G at 0 vanishes, whence for the Hamiltonian flow $\mathbf{G} : D^2 \times \mathbb{R} \rightarrow D^2$ of g the corresponding tangent map $T_0\mathbf{G}_t : T_0D^2 \rightarrow T_0D^2$ is always the identity. Hence for any C^∞ function α the tangent map at 0 of \mathbf{G}_α is the identity as well. Therefore for $t \neq \pi n$, $n \in \mathbb{Z}$, then $\mathbf{F}_t \neq \mathbf{G}_\alpha$ for any $\alpha \in C^\infty(M, \mathbb{R})$.

By Theorem 3.0.3 the shift map $\varphi_F : \mathcal{Z}_\omega^\mathbb{R}(f) \rightarrow \text{Stab}_0(f, \omega)$ of F is an infinite cyclic covering, while the shift map

$$\varphi_G : \mathcal{Z}_\omega^\mathbb{R}(g) \equiv \mathcal{Z}_\omega^\mathbb{R}(f) \longrightarrow \text{Stab}_0(f, \omega) \equiv \text{Stab}_0(g, \omega)$$

of G turns out to be not surjective, since its image does not contain \mathbf{F}_t for $t \neq \pi n$, $n \in \mathbb{Z}$.

Thus we see that the centralizer of g does not “detect” all the diffeomorphisms from $\text{Stab}_0(g)$, while the centralizer of f does so. This shows that the assumption $f \in \mathcal{F}(M, P)$ in Theorem 3.0.3 is essential.

3.2. Kronrod-Reeb graph of f . Now we will give an interpretation of \mathcal{Z} in terms of functions on the Kronrod-Reeb graph of f .

Let $f \in \mathcal{F}(M, P)$. Consider the partition Δ of M into connected components of level-sets of f . Let $K := M/\Delta$ be the corresponding quotient space and $p : M \rightarrow K$ be the factor map. Then we have a natural decomposition

$$f = \hat{f} \circ p : M \xrightarrow{p} K \xrightarrow{\hat{f}} P.$$

Endow K with the final topology, so a subset $U \subset K$ is open if and only if $p^{-1}(U)$ is open in M . Then it is well known that K has a natural structure of a one-dimensional CW-complex. It is called a *Lyapunov* or *Kronrod-Reeb* graph of f , [1], [22], [6], [5], [3].

We will briefly recall the correspondence between elements of Δ (i.e. points of K) and orbits of H . Let $\gamma \in \Delta$. If γ contains at least one critical point of f , then it follows from Axiom (L) that γ is a connected 1-dimensional CW-complex such that each of its vertices has even (possibly zero) degree, and $p(\gamma)$ is a vertex of K . In this case the vertices of γ are critical points of f being also zeros of H , while edges of γ are non-closed orbits of H .

If γ has no critical point of f , then γ is a closed orbit of H .

3.2.1. Lemma. *Each $\alpha \in \mathcal{Z}_\omega^Q(f)$ yields a unique continuous function $\hat{\alpha} : K \rightarrow Q$ such that $\alpha = \hat{\alpha} \circ p$. Moreover, the correspondence $\alpha \mapsto \hat{\alpha}$ is a continuous injective map $\eta : \mathcal{Z}_\omega^Q(f) \rightarrow C(K, Q)$ with respect to C^0 topology on $C(K, Q)$.*

Proof. Let $\alpha \in \mathcal{Z}_\omega^Q(f)$, so α takes constant values along orbits of H . First we should show that α takes constant value at each element of Δ .

Consider any element $\gamma \in \Delta$. If γ contains no critical point of f , then γ is a closed orbit of H , and so α takes a constant value at γ , see Figure 3.1.

Otherwise, γ is a connected 1-dimensional CW-complex whose vertices and edges are orbits of H . Then α takes constant values along edges of γ , and it follows from continuity of α and connectedness of γ that α is constant on all of γ .

Thus α yields a unique function $\hat{\alpha} : K \rightarrow Q$ such that $\alpha = \hat{\alpha} \circ p$.

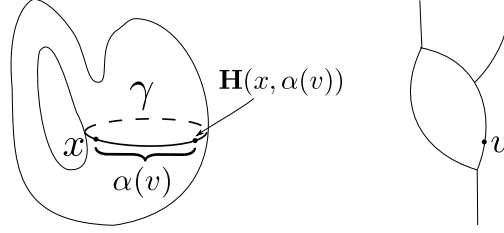


FIGURE 3.1.

Since K has final topology with respect to p and α is continuous, it easily follows and is well known that $\hat{\alpha}$ is continuous. Continuity of the correspondence $\alpha \mapsto \hat{\alpha}$ is left for the reader. \square

Lemma 3.2.1 together with Theorem 3.0.3 implies that for $f \in \mathcal{F}(M, P)$ the elements of $\text{Stab}_0(f, \omega)$ are parametrized by continuous functions on the Kronrod-Reeb graph K of f , see Figure 3.1.

More precisely, due to Theorem 3.0.3 for each $h \in \text{Stab}_0(f, \omega)$ there exists $\alpha \in \mathcal{Z}_\omega^Q(f)$ such that $h = \mathbf{H}_\alpha$. This function takes constant values on connected components of level-sets of f , and therefore induces a continuous function $\hat{\alpha} : K \rightarrow \mathbb{R}$. Then the value of $\hat{\alpha}$ at some point $v \in K$ equals to the common time shift induced by h on all the orbits of H constituting $p^{-1}(v)$.

3.3. Non-surjectivity of the map $j_0 : \pi_0 \text{Stab}(f, \omega) \rightarrow \pi_0 \text{Stab}^+(f)$. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$g(x, y) = ((x+1)^2 + y^2)((x-1)^2 + y^2).$$

It has three critical points: one saddle $p_0 = (0, 0)$ and two local minimums $p_1 = (-1, 0)$ and $p_2 = (1, 0)$. Let $D = g^{-1}[0, 2]$, and $f = g|_D : D \rightarrow \mathbb{R}$ be the restriction of g to D . Then D is a 2-disk and f belongs to the class $\mathcal{F}(D, \mathbb{R})$.

Consider the following subset $A = f^{-1}[0, 0.5] \subset D$, see Figure 3.2. It consists of two connected components A_1 and A_2 containing the points p_1 and p_2 respectively. Notice that

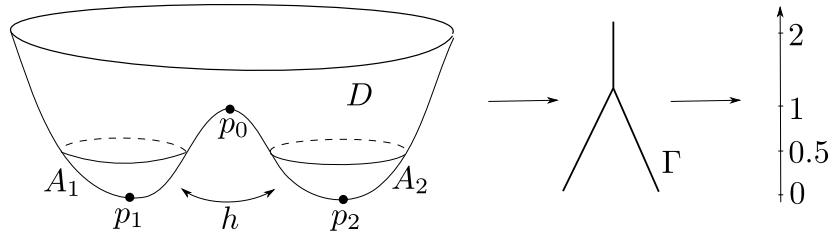


FIGURE 3.2.

$h(A) = A$ for each $h \in \text{Stab}(f)$, whence h either preserves both A_1 and A_2 or interchanges them. Also notice that if $h, k \in \text{Stab}(f)$ and $h(A_1) = A_2$, while $k(A_1) = A_1$, then h and k belong to distinct path components of $\text{Stab}(f)$.

Let $h : D \rightarrow D$ be a diffeomorphism defined by $D(x, y) = (-x, -y)$. Evidently, h belongs to $\text{Stab}^+(f)$ and interchanges A_1 and A_2 .

3.3.1. Lemma. *Let ω be any volume form on D such that $\text{Vol}_\omega(A_1) \neq \text{Vol}_\omega(A_2)$. Then the isotopy class $[h] \in \pi_0 \text{Stab}^+(f)$ of h does not contain any $k \in \text{Stab}(f, \omega)$. Hence for such an ω the map $j_0 : \pi_0 \text{Stab}(f, \omega) \rightarrow \pi_0 \text{Stab}^+(f)$ is not surjective.*

Proof. Each $k \in \text{Stab}(f, \omega)$ preserves ω -volume. Since $\text{Vol}_\omega(A_1) \neq \text{Vol}_\omega(A_2)$, it follows that $k(A_i) = A_i$ for $i = 1, 2$. But $h(A_1) = A_2$, whence h and k are not isotopic in $\text{Stab}(f)$. \square

REFERENCES

- [1] G. M. Adelson-Welsky and A. S. Kronrod. Sur les lignes de niveau des fonctions continues possédant des dérivées partielles. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 49:235–237, 1945.
- [2] Augustin Banyaga. Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. *Comment. Math. Helv.*, 53(2):174–227, 1978.
- [3] A. V. Bolsinov and A. T. Fomenko. *Vvedenie v topologiyu integriruemyykh gamiltonovykh sistem (Introduction to the topology of integrable Hamiltonian systems)*. “Nauka”, Moscow, 1997.
- [4] M. Dehn. Die Gruppe der Abbildungsklassen. *Acta Mathematica*, 69:135–206, 1938.
- [5] John Franks. Nonsingular Smale flows on S^3 . *Topology*, 24(3):265–282, 1985.
- [6] A. S. Kronrod. On functions of two variables. *Uspehi Matem. Nauk (N.S.)*, 5(1(35)):24–134, 1950.
- [7] E. A. Kudryavtseva. The topology of spaces of Morse functions on surfaces. *Math. Notes*, 92(1-2):219–236, 2012. Translation of *Mat. Zametki* **92** (2012), no. 2, 241–261.
- [8] E. A. Kudryavtseva. On the homotopy type of spaces of Morse functions on surfaces. *Mat. Sb.*, 204(1):79–118, 2013.
- [9] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Proc. Cambridge Philos. Soc.*, 60:769–778, 1964.
- [10] Sergiy Maksymenko. Smooth shifts along trajectories of flows. *Topology Appl.*, 130(2):183–204, 2003.
- [11] Sergiy Maksymenko. Homotopy types of stabilizers and orbits of Morse functions on surfaces. *Ann. Global Anal. Geom.*, 29(3):241–285, 2006.
- [12] Sergiy Maksymenko. Homotopy dimension of orbits of Morse functions on surfaces. *Travaux Mathématiques*, 18:39–44, 2008.
- [13] Sergiy Maksymenko. ∞ -jets of diffeomorphisms preserving orbits of vector fields. *Cent. Eur. J. Math.*, 7(2):272–298, 2009.
- [14] Sergiy Maksymenko. Reparametrization of vector fields and their shift maps. *Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos.*, 6(2):489–498, arXiv:math/0907.0354, 2009.
- [15] Sergiy Maksymenko. Functions with isolated singularities on surfaces. *Geometry and topology of functions on manifolds. Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos.*, 7(4):7–66, 2010.
- [16] Sergiy Maksymenko. Local inverses of shift maps along orbits of flows. *Osaka Journal of Mathematics*, 48(2):415–455, 2011.
- [17] Sergiy Maksymenko. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. page arXiv:math/1311.3347v3, 2016.
- [18] Dusa McDuff. Remarks on the homotopy type of groups of symplectic diffeomorphisms. *Proc. Amer. Math. Soc.*, 94(2):348–352, 1985.
- [19] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.
- [20] Jürgen Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.*, 120:286–294, 1965.
- [21] Leonid Polterovich. *The geometry of the group of symplectic diffeomorphisms*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001.
- [22] Georges Reeb. *Sur certaines propriétés topologiques des variétés feuilletées*. Actualités Sci. Ind., no. 1183. Hermann & Cie., Paris, 1952. Publ. Inst. Math. Univ. Strasbourg 11, pp. 5–89, 155–156.

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